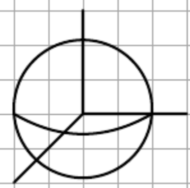
Q: Suppose a sphere has radius r and is centered at origin. what's its volumn?

Sol1:



$x^2+y^2+z^2=\alpha^2$

we seek $Vol(S)$ as a double integral

For Height function: $z^2=\alpha^2-x^2-y^2$

so upper hemisphere is $z=\sqrt{\alpha^2-x^2-y^2}$

so Lower hemisphere is $z=-\sqrt{\alpha^2-x^2-y^2}$

total height = upper − lower

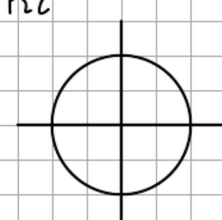
$$h(x,y)=\sqrt{\alpha^2-x^2-y^2}-\left(-\sqrt{\alpha^2-x^2-y^2}\right)$$
$$=2\sqrt{\alpha^2-x^2-y^2}$$

To parameterize R_r, we wrote

$$R_r=\{(x,y): x^2+y^2\le\alpha^2\}$$
$$=\{(x,y): -\alpha\le x\le\alpha,\ -\sqrt{\alpha^2-x^2}\le y\le\sqrt{\alpha^2-x^2}\}$$

$$\therefore Vol(S)=\iint_R h(x,y)\,dA$$
$$=\int_{x=-\alpha}^{\alpha}\int_{-\sqrt{\alpha^2-x^2}}^{\sqrt{\alpha^2-x^2}}\left(2\sqrt{a^2-x^2-y^2}\right)dy\,dx$$

Pic



$x^2+y^2=r^2$

$y=\pm\sqrt{r^2-x^2}$

Inner Integral:

$$\int 2\left[(\alpha^2-x^2)-y^2\right]^{\frac{1}{2}}dy$$

$$=2\int\sqrt{\alpha^2-x^2}\cos\theta\cdot\sqrt{\alpha^2-x^2}\cos\theta\,d\theta$$

$$=2(\alpha^2-x^2)\cdot\int\cos^2\theta\,d\theta$$

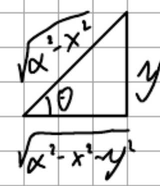
$$=(\alpha^2-x^2)\cdot\int 1-\cos(2\theta)d\theta$$

$$=(\alpha^2-x^2)\cdot(\theta+\tfrac{1}{2}\sin(2\theta)+C$$

$$=(\alpha^2-x^2)\cdot(\theta+\sin\theta\cos\theta)+C$$

$$=(\alpha^2-x^2)\cdot\left(\sin^{-1}\left(\frac{y}{\sqrt{\alpha^2-x^2}}\right)+\frac{y}{\sqrt{\alpha^2-x^2}}\cdot\frac{\sqrt{\alpha^2-x^2-y^2}}{\sqrt{\alpha^2-x^2}}\right)+C$$

$$=(\alpha^2-x^2)\cdot\left(\sin^{-1}\left(\frac{y}{\sqrt{\alpha^2-x^2}}\right)+y\cdot\sqrt{\alpha^2-x^2-y^2}\right)+C$$



$$\frac{y}{\sqrt{\alpha^2-x^2}}=\sin\theta$$

$$y=\sin\theta\cdot\sqrt{\alpha^2-x^2}$$

$$\sqrt{r^2-x^2-y^2}=\sqrt{\alpha^2-x^2}\cos\theta$$

$$dy=\sqrt{\alpha^2-x^2}\cos\theta\,d\theta$$

$\therefore$ Evaluating we obtain

$$\int_{y=-\sqrt{\alpha^2-x^2}}^{\sqrt{\alpha^2-x^2}}2\sqrt{\alpha^2-x^2-y^2}\,dy$$

$$=\left[(\alpha^2-x^2)\cdot\left(\sin^{-1}\left(\frac{y}{\sqrt{\alpha^2-x^2}}\right)+y\cdot\sqrt{\alpha^2-x^2-y^2}\right)+C\right]_{-\sqrt{\alpha^2-x^2}}^{\sqrt{\alpha^2-x^2}}$$

$$=\left[(\alpha^2-x^2)\cdot\sin^{-1}(1)+\sqrt{\alpha^2-x^2}\cdot\sqrt{0}\right]-\left[(\alpha^2-x^2)\cdot\sin^{-1}(-1)+\sqrt{\alpha^2-x^2}\cdot\sqrt{0}\right]$$

$$=(\alpha^2-x^2)\cdot\left[\sin^{-1}(1)-\sin^{-1}(-1)\right]$$

$$=(\alpha^2-x^2)\cdot\left(\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right)$$

$$=\pi(\alpha^2-x^2)$$

Outer integral

$$\int_{x=-\alpha}^{\alpha} \pi(\alpha^2 - x^2)\, dx$$

$$= \pi \left[r^2 x - \frac{1}{3}x^3 \right]_{-\alpha}^{\alpha}$$

$$= \pi \left[(\alpha^3 - \frac{1}{3}\alpha^3) - (-\alpha^3 + \frac{1}{3}\alpha^3) \right]$$

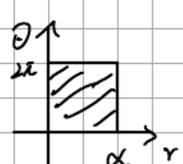
$$= \pi \left[2\alpha^3 - \frac{2}{3}\alpha^3 \right]$$

$$= \frac{4}{3}\pi\alpha^3$$

NB: That was computationally complicated

If we use polar coordinates for integral, the region and height function are much simplier

$$R_{polar} = \{ (r, \theta),\ 0 \le r \le \alpha,\ 0 \le \theta \le 2\pi \}$$

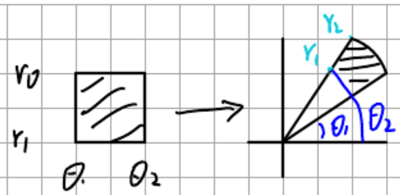


$$h(x,y) = 2\sqrt{\alpha^2 - x^2 - y^2}$$

$$\Downarrow$$

$$h(r\cos\theta, r\sin\theta) = 2\sqrt{\alpha^2 - (r^2\cos^2\theta + r^2\sin^2\theta)} = 2\sqrt{\alpha^2 - r^2}$$

To consider the differential, consider a small rectangle In the cartesian plane corresponds to a circular plane



Area of a (small) circular section is:

$$\frac{1}{2} r_2^2 (\theta_2 - \theta_1) - \frac{1}{2} r_1^2 (\theta_2 - \theta_1)$$

$$= \frac{1}{2} (r_2^2 - r_1^2)(\theta_2 - \theta_1)$$

$$= \frac{1}{2} (r_2 + r_1)(r_2 - r_1)(\theta_2 - \theta_1)$$

$$\therefore \Delta A = \frac{1}{2}(r_2 + r_1) \cdot \Delta r \cdot \Delta\theta = \frac{1}{2}(r_2 + r_1) \cdot \Delta A_{polar}$$

Now limiting as $\Delta A \to 0$ $(\Delta\theta \to 0, \Delta r \to 0)$

We see $\frac{1}{2}(r_2 + r_1) \to \frac{1}{2} 2r^* = r^*$

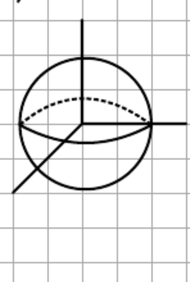
Hence in the limit we obtain

$$dA_{cart} = r\, dA_{polar}$$

Volumn of Sphere:

Sol 2: (with polar coordinates):

In polar coordinates (i.e. (r, θ, z) plane)



$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$z = \sqrt{\alpha^2 - x^2 - y^2}$$

$$= \sqrt{\alpha^2 - r^2}$$

$$R_{polar} = [0, 2\pi] \times [0, \alpha]$$

$$Vol(S) = \iint_{R_{cart}} h(x,y)\ dA_{cart}$$

$$= \iint_{R_{polar}} h(r\cos\theta, r\sin\theta) \cdot r\, dA_{polar}$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{\alpha} 2\sqrt{\alpha^2 - r^2} \cdot r\, dr\, d\theta$$

Inner Integral:

$$\int_{r=0}^{r=\alpha} 2r\sqrt{\alpha^2-r^2}\ dr \qquad u = a^2 - r^2$$
$$du = -2r\,dr$$

$$= \int_{r=0}^{\alpha} u^{\frac{1}{2}}\ du$$

$$= \left[-\frac{2}{3}u^{\frac{3}{2}}\right]_0^{\alpha}$$

$$= \left[-\frac{2}{3}(\alpha^2 - r^2)^{\frac{3}{2}}\right]_0^{\alpha}$$

$$= -\frac{2}{3}(\alpha^2-\alpha^2)^{\frac{3}{2}} - \left(-\frac{2}{3}(\alpha^2-0)^{\frac{3}{2}}\right)$$

$$= \frac{2}{3}\alpha^3$$

Outer Integral:

$$\int_0^{2\pi} \frac{2}{3}\alpha^3\, d\theta$$

$$= \frac{2}{3}\alpha^3 \int_0^{2\pi} d\theta$$

$$= \frac{2}{3}\alpha^3 [\theta]_0^{2\pi}$$

$$= \frac{4}{3}\pi\alpha^3$$

Ex. compute $\iint_R \sin(\sqrt{x^2+y^2})\,dA$ for the region bounded by $x^2+y^2=1$, $x^2+y^2=9$

Sol: Turn the integral into polar form

$R_{polar} = \{(r,\theta) : 1 \le r \le 3,\ 0 \le \theta \le 2\pi\}$

$f(x,y) = \sin(\sqrt{x^2+y^2})$ has polar form

$f(r\cos\theta, r\sin\theta) = \sin(\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}) = \sin(r)$

$\therefore \iint_{R_{cart}} \sin(\sqrt{x^2+y^2})\,dA_{cart}$

$= \iint_{R_{polar}} \sin(r)\cdot r\,dA_{polar}$

$= \int_{\theta=0}^{2\pi}\int_{r=1}^{3} r\cdot\sin(r)\,dr\,d\theta$ $\quad$ $u = r \quad dv = \sin(r)$, $\quad du = dr \quad v = -\cos(r)$

$= \int_0^{2\pi} \left[-r\cos(r) - \int -\cos(r)\,dr\right]_1^3 d\theta$

$= \int_0^{2\pi} \left[-r\cos(r) + \sin(r)\right]_1^3 d\theta$

$= \int_0^{2\pi} (-3\cos(3) + \sin(3)) - (-\cos(1) + \sin(1))\,d\theta$

$= (\sin(3) - \sin(1) - 3\cos(3) + \cos(1)) \cdot [\theta]_0^{2\pi}$

$= 2\pi\cdot(\sin(3) - \sin(1) - 3\cos(3) + \cos(1))$

Exercise: Compute $\iint_R x\cdot\exp(-x^2-y^2)\,dA$ on R the disk of radius 3 about the origin.